

Quasi-instantons in QCD with chiral symmetry restoration

Takuya Kanazawa¹ and Naoki Yamamoto²

¹*THES Research Group and Quantum Hadron Physics Laboratory, RIKEN, Wako, Saitama 351-0198, Japan*

²*Department of Physics, Keio University, Yokohama 223-8522, Japan*

We show, without using semiclassical approximations, that, in high-temperature QCD with chiral symmetry restoration and $U(1)_A$ symmetry breaking, the partition function for sufficiently light quarks can be expressed as an ensemble of noninteracting objects with topological charge that obey the Poisson statistics. We argue that the topological objects are “quasi-instantons” (rather than bare instantons) taking into account quantum effects. Our result is valid even close to the (pseudo)critical temperature of the chiral phase transition.

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I. INTRODUCTION

The instantons [1, 2], which are topological excitations of QCD in Euclidean spacetime, play key roles in non-perturbative aspects of QCD. In particular, they solve the $U(1)_A$ puzzle [3, 4] and provide us with a semiclassical picture of chiral symmetry breaking in the QCD vacuum [2]. The instanton liquid model [2] is proposed to describe the physically relevant regimes at finite temperature and density that are accessible in heavy ion collision experiments at RHIC/LHC. However, model-independent many-body instanton descriptions based on QCD are reliable only in some extreme cases: QCD at sufficiently high temperature, $T \gg T_c$ with T_c being the (pseudo)critical temperature of chiral transition [1, 5, 6], or at sufficiently large chemical potential, $\mu \gg \Lambda_{\text{QCD}}$ [7–9]. This is thanks to the presence of small expansion parameters, $\Lambda_{\text{QCD}}/T \ll 1$ and $\Lambda_{\text{QCD}}/\mu \ll 1$, but it is not the case for $T \sim \Lambda_{\text{QCD}}$ and/or $\mu \sim \Lambda_{\text{QCD}}$. This is one of the obstacles that prevents us from understanding the properties of physically interesting phases at finite T and/or μ on the basis of QCD itself.

In this paper, we show that QCD at $T > T_c$, even close to T_c , can be seen as an ensemble of *noninteracting quasi-instantons*, which could be seen as “dressed” quasiparticles in the strongly interacting ensemble of conventional instantons (calorons). We demonstrate this picture in a theoretically controlled manner as follows¹: we start with the effective theory in terms of the quark mass M

that includes the dependence on the θ angle and quantum fluctuations systematically. We then show that the partition function of this effective theory can be understood as an ensemble of noninteracting objects with positive or negative topological charges which obey the Poisson distribution. We shall refer to such objects as *(anti-)quasi-instantons*.

Our approach should be contrasted with the conventional instanton analysis where one adds quantum effects to the classical instantons step by step. In our approach, quantum effects are already included in the effective theory, and then we rewrite it in a physically different form in terms of the quasi-instantons. In particular, we do *not* use the semiclassical approximation in this approach. While this procedure looks similar to the ones [7, 9] considered in the color-superconducting phases of QCD at zero temperature and high density, the starting effective theories and expansion parameters are completely different.

For simplicity, we first consider $N_f = 2$ as an example, and later extend it to general $N_f \geq 2$. Our argument is also applicable to QCD at finite chemical potential μ and/or in a magnetic field B in a straightforward manner, as long as the chiral symmetry of QCD, except $U(1)_A$, is restored.

II. QCD PARTITION FUNCTION

Let us consider the QCD partition function as a function of quark mass M . Here we consider the high-temperature regime $T > T_c$ and expand the free energy of QCD in terms of a small parameter $m/T \lesssim m/\Lambda_{\text{QCD}} \ll 1$. Recalling that quarks acquire the effective thermal mass and there are no massless Nambu-Goldstone modes at $T > T_c$, the free energy of the system is analytic in

¹ We note that the main assumption in this paper is the breaking of $U(1)_A$ symmetry through the axial anomaly [or precisely speaking, $f_A \neq 0$ in Eq. (4)]. This is shown in QCD at $T \gg T_c$ in Refs. [1, 5, 6] through controlled semiclassical calculations (see also Ref. [10]), and it seems natural to expect $f_A \neq 0$ at any $T > T_c$ (see, however, Refs. [11–15]).

quark mass.

In order to write down the general form of the free energy in terms of M , we use the spurion field method. We allow the quark mass matrix M to transform under the symmetry, $\mathcal{G} \equiv \text{SU}(N_f)_R \times \text{SU}(N_f)_L \times \text{U}(1)_A$, so that the mass term in the original QCD Lagrangian,

$$\mathcal{L}_{\text{mass}} = \psi_L^\dagger M \psi_R + \text{h.c.}, \quad (1)$$

is invariant under \mathcal{G} , where $\psi_{R,L}$ are the right- and left-handed quarks. As $\psi_{R,L}$ transform under \mathcal{G} as

$$\psi_R \rightarrow e^{i\theta_A} V_R \psi_R, \quad \psi_L \rightarrow e^{-i\theta_A} V_L \psi_L, \quad (2)$$

with θ_A being the $\text{U}(1)_A$ phase, we impose the following transformation law for M :

$$M \rightarrow e^{-2i\theta_A} V_L M V_R^\dagger. \quad (3)$$

Using Eq. (3), one can construct the general free energy density of two-flavor QCD invariant under $\text{SU}(2)_R \times \text{SU}(2)_L$ but *not* under $\text{U}(1)_A$, as

$$f = f_0 - f_2 \text{tr} M^\dagger M - f_A (\det M + \det M^\dagger) + O(M^4). \quad (4)$$

Here f_0 , f_2 , and f_A are parameters that are functions of T and V_3 (the spatial volume) [16]. The term $i(\det M - \det M^\dagger)$, which breaks parity ($M \leftrightarrow M^\dagger$), is omitted. While the f_2 term is chirally symmetric and invariant under $\text{U}(1)_A$, the f_A term is *not* invariant under $\text{U}(1)_A$ and expresses the QCD axial anomaly. One microscopic explanation for the f_A term may be given by the (bare) instanton-induced interactions [3], but our discussion below does not depend on the specific origin of f_A ; the main assumption we make is $f_A \neq 0$.

Superficially, Eq. (4) might look similar to the Ginzburg-Landau theory, where the free energy is expanded in terms of a small order parameter around a phase transition based on symmetries. In Eq. (4), on the other hand, high-order terms in M are suppressed in terms of the small parameter $m/T \ll 1$, and the expansion is *not* limited to the region near the chiral transition; also, M is not an order parameter for some symmetry.²

² In the case of a second-order chiral phase transition, one needs to let $m \rightarrow 0$ in the limit $T \rightarrow T_c$ in order to keep the mass expansion (4) convergent. This is because the coefficients in Eq. (4) tend to diverge as $T \rightarrow T_c$, which reflects the nonanalytic mass dependence of f at $T = T_c$.

III. THE θ ANGLE AND TOPOLOGICAL SUSCEPTIBILITY

One can incorporate the dependence of the θ angle in Eq. (4) via the replacement, $M \rightarrow M e^{i\theta/N_f}$. This is because, for

$$Z(\theta) = \sum_{q=-\infty}^{\infty} e^{iq\theta} Z_q, \quad (5)$$

the zero-mode contribution to the fermion determinant in Z_q is given by $(\det M)^q$ for $q \geq 0$ and $(\det M^\dagger)^{-q}$ for $q < 0$, and so $Z(\theta)$ depends on θ only through the combination $M e^{i\theta/N_f}$ [17]. In two-flavor QCD with $M = \text{diag}(m_u, m_d)$, the resulting free energy is

$$f(\theta) = \tilde{f} - 2f_A m_u m_d \cos \theta + O(m^4), \quad (6)$$

where we define $\tilde{f} = f_0 - f_2(m_u^2 + m_d^2)$.

With this free energy density $f(\theta)$, the partition function is given by

$$Z_{\text{QCD}}(\theta) = \exp[-V_4 f(\theta)], \quad (7)$$

where $V_4 \equiv V_3/T$ is the four-volume. It is then easy to derive the topological susceptibility,

$$\chi \equiv -\frac{1}{V_4} \left. \frac{\partial^2 \ln Z_{\text{QCD}}}{\partial \theta^2} \right|_{\theta=0} = 2f_A m_u m_d + O(m^4). \quad (8)$$

This should be contrasted with the topological susceptibility in the QCD vacuum, $\chi_{\text{vac}} = \Sigma(m_u^{-1} + m_d^{-1})^{-1}$, with Σ being the magnitude of the chiral condensate [17]. This difference is because the free energy of the QCD vacuum has an $O(M)$ term due to the spontaneous chiral symmetry breaking, whereas the free energy at $T > T_c$ does not [see Eq. (4)].

Equation (8) is consistent with the anomalous Ward identity derived by Veneziano [18], which for general N_f and for equal masses states that

$$\chi = -\frac{m}{N_f^2} \langle \bar{\psi}_f \psi_f \rangle + \frac{m^2}{N_f^2} \int d^4x \langle \bar{\psi}_f \gamma_5 \psi_f(x) \bar{\psi}_g \gamma_5 \psi_g(0) \rangle, \quad (9)$$

where the sum over repeated indices is understood ($f, g = 1, \dots, N_f$). Indeed, for two-flavor QCD ($m_u = m_d = m$), substituting $\langle \bar{\psi}_f \psi_f \rangle = -4(f_2 + f_A)m + O(m^3)$ and $\int d^4x \langle \bar{\psi}_f \gamma_5 \psi_f(x) \bar{\psi}_g \gamma_5 \psi_g(0) \rangle = 4(f_A - f_2) + O(m^2)$ that follow from Eq. (4) into the rhs of Eq. (9), one gets $\chi = 2f_A m^2 + O(m^4)$, in agreement with Eq. (8). However, our direct derivation of the simple expression (8) from the expansion of the QCD free energy at $T > T_c$ is new. Within our approach, not only the topological susceptibility but also the higher-order moments of the

winding number q can be derived in a straightforward manner, by taking derivatives of $\ln Z_{\text{QCD}}$ with θ at $\theta = 0$ as

$$\langle q^2 \rangle = \mathcal{A}, \quad (10a)$$

$$\langle q^4 \rangle = \mathcal{A}(1 + 3\mathcal{A}), \quad (10b)$$

$$\langle q^6 \rangle = \mathcal{A}(1 + 15\mathcal{A} + 15\mathcal{A}^2), \quad (10c)$$

$$\langle q^8 \rangle = \mathcal{A}(1 + 63\mathcal{A} + 210\mathcal{A}^2 + 105\mathcal{A}^3), \quad (10d)$$

with $\mathcal{A} \equiv 2V_4 f_A m^2$, while all odd moments vanish. (We will derive the results for general N_f later.) If \mathcal{A} is fixed in the limit $m \rightarrow 0$ and $V_4 \rightarrow \infty$, then all the other θ -dependent terms in $\ln Z_{\text{QCD}}(\theta)$ drop off and these results become *exact*.

We emphasize that the arguments above are based only on the symmetry and analyticity of QCD under the systematic expansion, and thus are fully under theoretical control. Note that, while analytical calculations of the θ -dependence are also possible at sufficiently high $T \gg \Lambda_{\text{QCD}}$ based on a dilute instanton gas picture [1], in the present paper we do not rely on the assumption of high T ; indeed, the above arguments are valid at generic $T > T_c$. Although the radius of convergence of the expansion (4) may vary with T , it does not affect our main results.

Equation (7) can be expressed as

$$\begin{aligned} Z_{\text{QCD}} &= e^{-V_4 \bar{f}} \exp \left(V_4 \lambda \sum_{Q=\pm 1} e^{iQ\theta} \right) \\ &= e^{-V_4 \bar{f}} \sum_{N=0}^{\infty} \frac{(V_4 \lambda)^N}{N!} \left(\sum_{Q=\pm 1} e^{iQ\theta} \right)^N, \end{aligned} \quad (11)$$

where $\lambda \equiv f_A m_u m_d$ and we used the Taylor expansion in the second line above. The $O(m^4)$ contribution is small and is disregarded in this expression. By using the identity

$$\begin{aligned} \left(\sum_{Q=\pm 1} e^{iQ\theta} \right)^N &= \sum_{Q_1=\pm 1} \cdots \sum_{Q_N=\pm 1} (e^{iQ_1\theta} \cdots e^{iQ_N\theta}) \\ &= \sum_{N_++N_-=N} \frac{N!}{N_+!N_-!} e^{iq\theta}, \end{aligned} \quad (12)$$

where $q \equiv \sum_{i=1}^N Q_i = N_+ - N_-$ with $N_{\pm} \geq 0$, we arrive at the expression

$$Z_{\text{QCD}} = e^{-V_4 \bar{f}} \sum_{N_+=0}^{\infty} \frac{(V_4 \lambda e^{i\theta})^{N_+}}{N_+!} \sum_{N_-=0}^{\infty} \frac{(V_4 \lambda e^{-i\theta})^{N_-}}{N_-!}. \quad (13)$$

This is the main equation of this paper. We will denote the piece in the summation in Eq. (13) as $g(N_{\pm}, \theta)$ for later convenience.

IV. QUASI-INSTANTON ENSEMBLE INTERPRETATION

To understand the physical meaning of Eq. (13), first recall that the θ angle enters the original QCD action only through the combination

$$S_{\theta} = -i\theta Q_{\text{top}}, \quad Q_{\text{top}} \equiv \frac{1}{32\pi^2} \int d^4x F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a. \quad (14)$$

In Eq. (12), on the other hand, the θ angle appears through the combination $iq\theta$, so one can make the identification $q = Q_{\text{top}}$. One can now understand that each $Q_i = \pm 1$ is to be identified with the integer topological charge. Hence Eq. (13) means that the QCD partition function can be written as an ensemble of N_{\pm} *noninteracting* objects that have positive or negative topological charges $Q_i = \pm 1$. We shall call them the (anti-)quasi-instantons, as they are different from the bare (anti-)instantons in that they are dressed with the classical and quantum effects of interactions.

This result may seem at first sight trivial due to the following reason: for a sufficiently small mass $m \ll T$, the statistical weight of an isolated instanton $\propto m^{N_f}$ is suppressed and instantons become dilute. Then the average distance between instantons is so large that they may well behave as effectively noninteracting objects. This picture is oversimplified, however, for the actual gauge field is not a dilute instanton gas (unless $T \gg T_c$), but rather a crowded mixture of (anti-)instantons overlapping with each other. In particular, the so-called instanton-anti-instanton molecules [2], which are topologically neutral and evade suppression by powers of m , will proliferate and call for a treatment as a strongly coupled many-body system. Therefore, it is a highly nontrivial finding that, despite nontrivial interactions between bare instantons (see, e.g., Ref. [2]), after integrating out the effects of interactions [which is indirectly done through Eq. (4), where the effects of interactions are included in the coefficients of the expansion], the resulting quasi-instantons turn out to be noninteracting. This is somewhat similar to the idea of Landau's Fermi liquid theory [19], where the system is described in terms of weakly interacting quasiparticles that have the same quantum numbers as the original particles but are dressed with the effects of interactions.

A similar identification of “quasi-instantons” in the two-flavor color superconductivity and in the color-flavor locked phase of QCD at high density were previously noted in Refs. [7] and [9], respectively. There, they arrive at the ensemble of *weakly interacting* quasi-instantons through a different path: the duality mapping of the

low-energy effective theories for the η and η' mesons, respectively. In contrast, the quasi-instantons here do not interact with each other, unlike Refs. [7, 9]. Physically, this difference is due to the fact that in hot QCD, there is no (pseudo-)Nambu-Goldstone mode associated with some symmetry breaking that mediates the interaction between (anti-)quasi-instantons in the present case.

V. MOMENTS AND POISSON DISTRIBUTION

Once the partition function is understood in terms of (anti-)quasi-instantons, it is easy to derive the density of (anti-)quasi-instantons, susceptibility, and higher moments, following Ref. [9]. For example, at $\theta = 0$ we have

$$\langle N_{\pm} \rangle = \frac{\exp(-V_4 \tilde{f})}{Z_{\text{QCD}}} \sum_{N_{\pm}=0}^{\infty} N_{\pm} g(N_{\pm}, 0) = V_4 \lambda, \quad (15)$$

with $\langle \dots \rangle$ the expectation value with respect to the QCD measure, and so the number density of (anti-)quasi-instantons is $n_{\pm} = \langle N_{\pm} \rangle / V_4 = \lambda = f_A m_u m_d$. Therefore, the average topological charge is $\langle Q \rangle = 0$, and the total quasi-instanton density is $n \equiv n_+ + n_- = 2\lambda$; the average spatial distance between quasi-instantons is $d_q \sim (f_A m^2 / T)^{-1/3}$.

By a procedure similar to the above, one can show [9]

$$\left\langle \frac{N_+!}{(N_+ - k)!} \frac{N_-!}{(N_- - l)!} \right\rangle = (\lambda V_4)^{k+l} \quad (16)$$

for any non-negative integer k and l . Recalling the property of Poisson statistics, $f(x) = e^{-\beta} \beta^x / x!$, that its n th factorial moment is β^n , we can conclude that the quasi-instantons and anti-quasi-instantons follow the Poisson statistics independently. Usually, the Poisson distribution for the ensemble of *bare* instantons and anti-instantons is expected only at $T \gg T_c$ [1]. Here, on the other hand, we have shown that the ensemble of *quasi*-instantons always obey the Poisson distribution at $T > T_c$, even for $T \sim T_c$,³ in a theoretically controlled manner. As $T \rightarrow \infty$, the quasi-instantons are expected to reduce to bare instantons, and f_A is estimated to be proportional to the one-instanton weight, $e^{-8\pi^2/g(T)^2} \ll 1$, with $g(T)$ the gauge coupling at the scale T .

³ The noninteracting quasi-instanton picture should remain valid if $d_q \gg \xi$, where ξ is the correlation length of the system. For a first-order transition, this condition is satisfied if $m \ll T$. For a second-order transition, $\xi \sim (T - T_c)^{-\nu}$ diverges, so one has to judiciously let $m \rightarrow 0$ near T_c . Note that this requirement has already appeared in footnote 2.

Note that higher-order terms with θ dependence in Eq. (4), e.g., $(\det M)^2 e^{2i\theta} + \text{h.c.}$, are related to multi-instanton effects, but are suppressed by some powers of $m_f / T_c \ll 1$. If m_f is not sufficiently small compared with T_c , such effects would become important and distort the Poisson distribution.

For the Poisson distribution, the susceptibility is equal to the first moment, and thus we have

$$\chi_{\text{inst}} = n_{\text{inst}} = 2\lambda, \quad (17)$$

which is equal to Eq. (8). This is valid for any $T > T_c$. From Eq. (16), higher moments in Eq. (10) can also be reproduced.

VI. EXTENSION

So far we have concentrated on $N_f = 2$. From now on, we generalize our argument to any $N_f \geq 2$. In this case, we use Eq. (3) again, but we need to modify the form of the free energy in Eq. (4). Considering all the possible terms to $O(M^{N_f})$ consistent with the symmetry \mathcal{G} , the free energy is given by

$$f(\theta) = F(M) - f_A (e^{i\theta} \det M + e^{-i\theta} \det M^\dagger) + O(M^{N_f+1}).$$

Here the only term involving the θ angle is the f_A term, and we denote all the other terms to $O(M^{N_f})$ by the $F(M)$ term. For $M = \text{diag}(m_1, m_2, \dots, m_{N_f})$, the free energy reads

$$f(\theta) = F(m_i) - 2f_A \cos \theta \prod_{i=1}^{N_f} m_i, \quad (18)$$

where $O(M^{N_f+1})$ terms have been neglected. Following the same steps as before, one gets the expression

$$Z_{\text{QCD}} = \exp(-V_4 F) \sum_{N_{\pm}=0}^{\infty} \frac{(V_4 \sigma)^{N_+ + N_-}}{N_+! N_-!} e^{i\theta(N_+ - N_-)}, \quad (19)$$

where $\sigma \equiv f_A \prod_{i=1}^{N_f} m_i$. One can also repeat the same interpretation in terms of (anti-)quasi-instantons as above, with the replacements $\tilde{f} \rightarrow F$ and $\lambda \rightarrow \sigma$. In particular, we have for the quasi-instanton density and the topological susceptibility

$$\chi_{\text{inst}} = n_{\text{inst}} = 2\sigma. \quad (20)$$

The higher moments are given in Eq. (10) with the replacement $\mathcal{A} \rightarrow 2V_4 \sigma$.

Finally, we comment on one-flavor QCD. In one-flavor QCD vacuum, the θ dependence of the free energy is $\sim \cos \theta$ [17], similarly to Eqs. (6) and (18). In this

case, one can expand the free energy density in terms of $m/\Lambda_{\text{QCD}} \ll 1$ as

$$f_{N_f=1} = f_0 - \Sigma m \cos \theta + O(m^2), \quad (21)$$

where Σ is the magnitude of the chiral condensate. One can reach the noninteracting quasi-instanton picture in this case as well. This is not limited to high temperature but holds true for *any* $T > 0$ because, for $N_f = 1$, chiral symmetry is always broken explicitly by the $U(1)_A$ anomaly [5].

VII. CONCLUSION

From the systematic effective theory that includes the θ dependence and quantum effects of QCD at $T > T_c$, we arrived at the simple picture of noninteracting (anti-)quasi-instantons for sufficiently light quarks. This picture may explain why the dilute instanton gas approximation, which has been conventionally expected to be valid only at sufficiently high temperature $T \gg T_c$, provides a reasonable description of QCD even close to T_c , as observed in lattice QCD simulations [20, 21] (see, however, Ref. [15]). A similar instanton gas behavior was also numerically observed for the pure $SU(N_c)$ (QCD with $m = \infty$) and G_2 lattice gauge theories just above T_c [22, 23]. This might imply that the noninteracting instanton picture at $T > T_c$ is valid independently of m , though it can be theoretically justified only for small m

from our argument at this moment.

Note that our argument above does not explicitly depend on T , except that chiral symmetry is restored at $T > T_c$. This implies that our argument and results are also applicable to dense and magnetized matter. The coefficients of the expansion in M then become functions of T , μ , and B [24]. We also stress that our argument so far works only in a chirally symmetric phase of QCD. In a phase with spontaneous chiral symmetry breaking, the θ dependence of the free energy is not like Eqs. (6) and (18) [17].

It would be an interesting question to study various correlation functions at $T > T_c$ in terms of the quasi-instanton picture. One may also be able to understand the spectral density of the Dirac operator in relation to (the breaking of) the $U(1)_A$ symmetry at $T > T_c$ in terms of the quasi-instanton picture. We will defer answering such questions to future work [25].

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